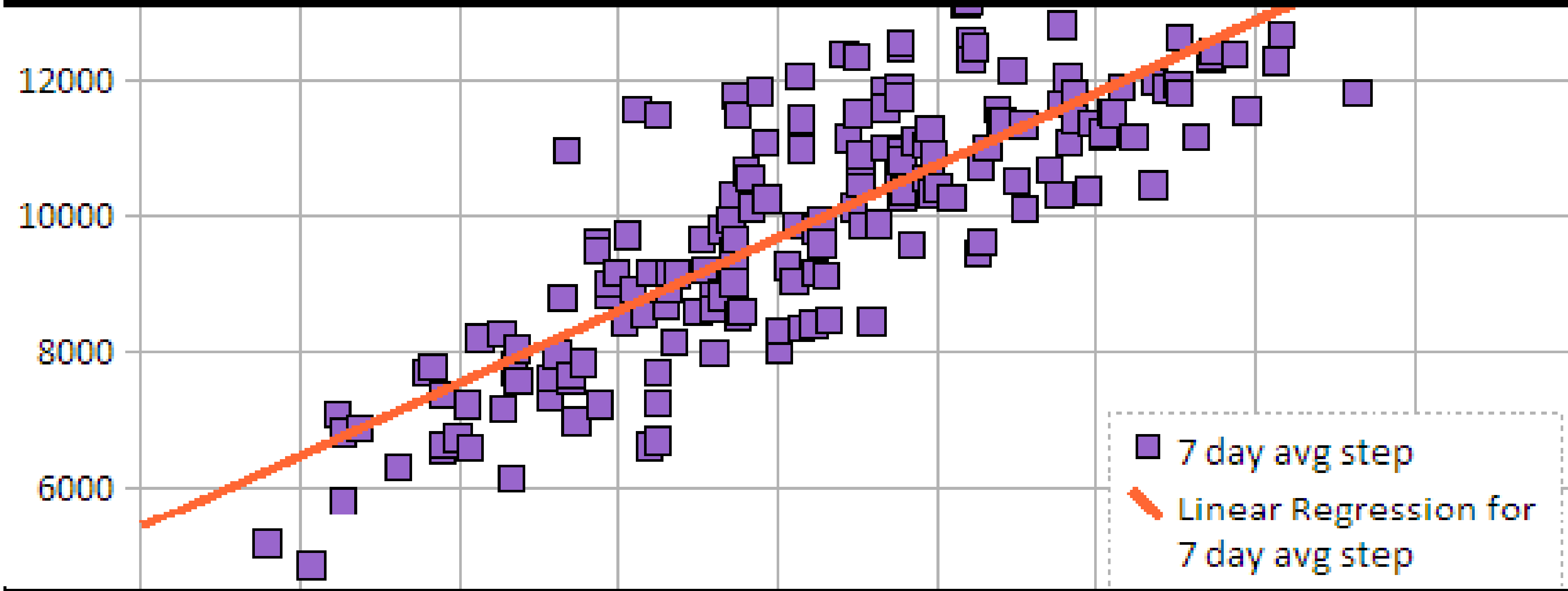
E80
Experimental Engineering

Lecture 2
Data Uncertainty, Data Fitting, Error Propagation



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Purpose & Outline

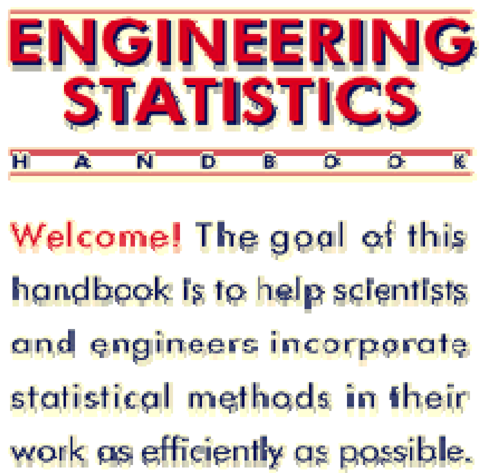
- Data Uncertainty & Confidence in Measurements
- Data Fitting - Linear Regression
- Error Propagation
- Quantization Error

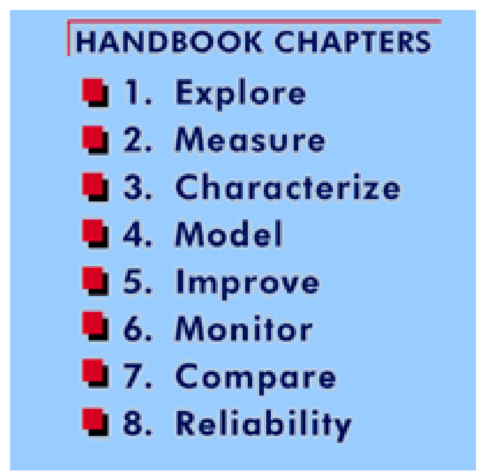
Context

- Understanding data uncertainty and being able to specify the confidence of measurements is a crucial engineering skill…
- Bad things happen when people do not understand or account for data uncertainty.
- From the Professional Engineering Practice Standpoint, understanding data uncertainty is a key **risk management** activity

Additional Resources

- "Data Analysis, Standard Error and Confidence Limits" supplemental handout on E80 website
- Engineering Statistics Handbook, NIST
  - http://www.itl.nist.gov/div898/handbook/index.htm
  - *ISO Guide to the Expression of Uncertainty in Measurement*





Outline

- Uncertainty & Confidence in Measurements
- Linear Regression
- Error Propagation
- Quantization Error

Confidence in Measurements

- When we take measurements, we want to know how "good" they are.

  "Uncertainty is a measure of the 'goodness' of a result. Without such a measure, it is impossible to judge the fitness of the value as a basis for making decisions relating to health, safety, commerce or scientific excellence" - NIST Engineering Statistics handbook

- Need to describe their "goodness" in a meaningful way

Basic Measurement Assumptions

- Each measurement (rocket motor mass, temperature, etc.) has some "random" noise & uncertainty
- There is some **"true"** value of x that we are trying to measure

$$\mu$$

- The distribution of measurements is normal (Gaussian) and the "True" value lies at the center of the distribution
- We will approximate μ and this distribution from our measurements

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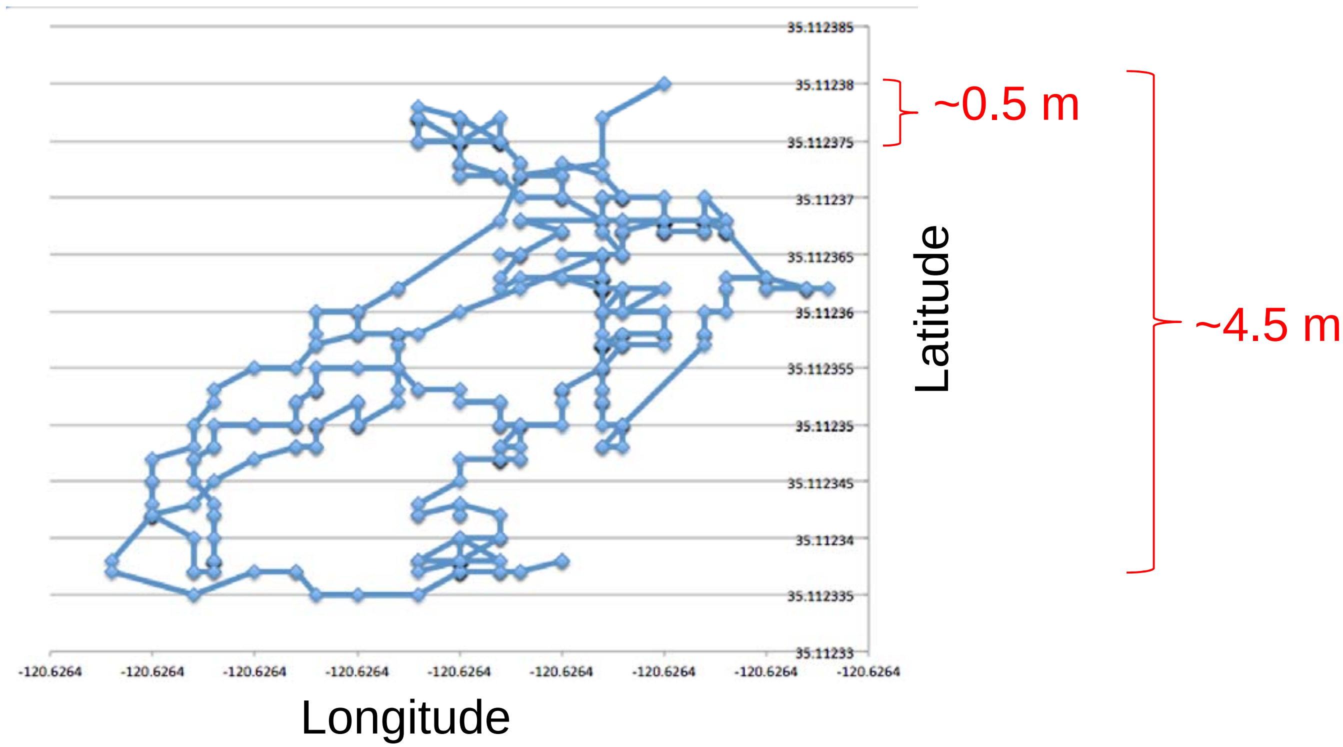
Example: What is the location?

- Professor Clark's AUV *stationary* GPS position output



For a basic measurement…

- Consider N **measurements**

$$x_1, x_2, x_3, \ldots x_N$$

Sample Mean & Error

- For N measurements, the **sample mean** is

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i ,$$

- If we knew the true value (μ), we could calculate the error in each measurement

$$\varepsilon = x_i - \mu$$

- We define the **residual error (residuals)**, for each measurement, to be

$$e_i = x_i - \bar{x}$$

The mean depends on the measurements… therefore only N-1 of the residuals are independent… Lost a degree of freedom

Sample Variance & Standard Deviation

- We characterize our residual errors using the **sample variance**:

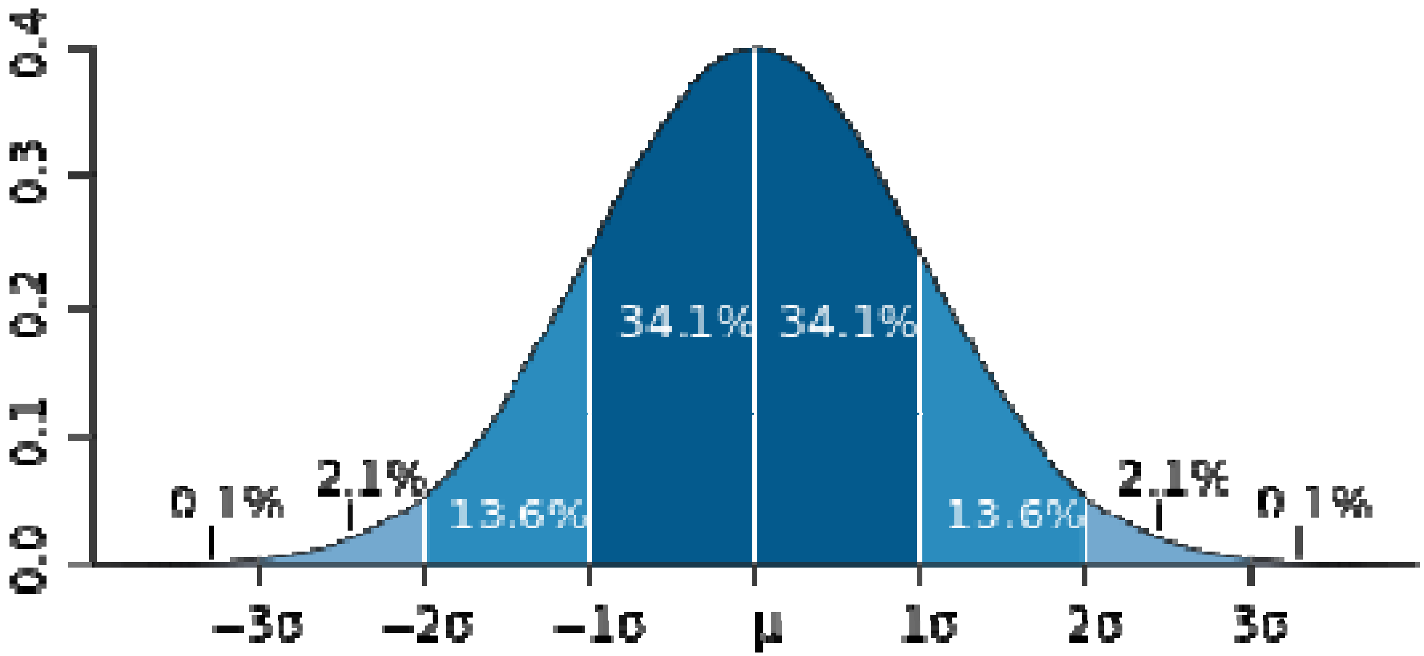
$$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N} e_i^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

  - The **sample variance** S^2 characterizes the spread of the measurements.

- The **sample standard deviation** can be defined as:

$$S = \sqrt{S^2}$$

  - S approximates σ (true std. dev.) → degree to which individual measurements x_i vary from μ, but does not tell us how far $\bar{x}$ is from μ



proportion of samples that would fall between 0, 1, 2, and 3 standard deviations above and below the actual value.

Estimated Standard Error

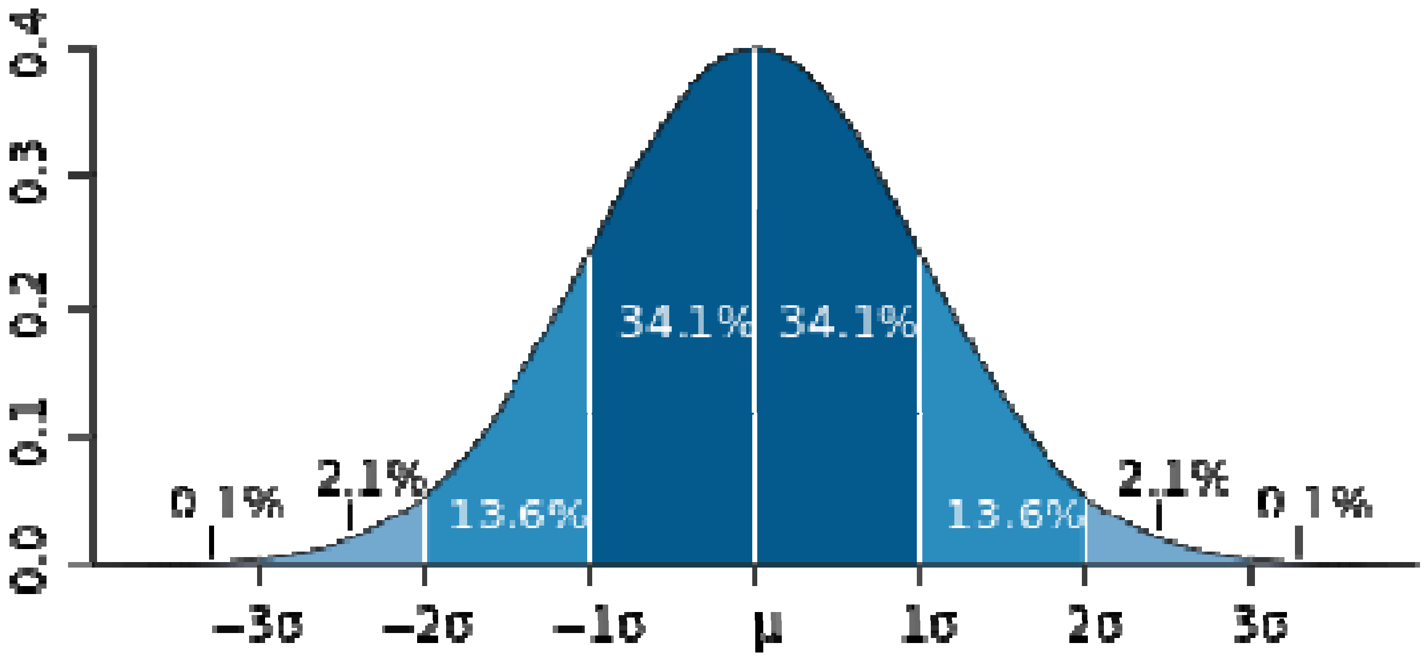
- We estimate how far the sample mean $\bar{x}$ is from the actual value μ using the **estimated standard error**:

$$ESE = \frac{S}{\sqrt{N}}$$

ex: sample mean x = 42.000, sample standard deviation S = 0.01, N = 200,

- ESE = 0.100/sqrt(200) = 0.0071
- $\bar{x}$ = 42.000 ± 0.007 (confidence level (λ) ?)

Standard Error Confidence Interval



$\bar{x} = 42.000 \pm 0.007$ (68% confidence interval)

Confidence Intervals

- As the sample size decreases, normal distribution under-reports uncertainty…
- The Student's T-Value (*t*) is used to estimate the confidence interval (λ)
  - relates the confidence interval to the area under a standard distribution

$$\lambda = ESE * t$$

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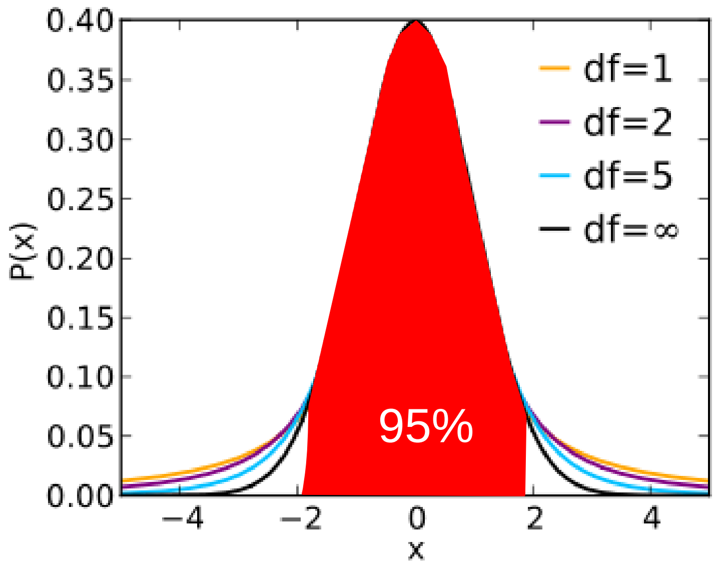
Student's T-Value

- Use lookup table to get t
  - confidence interval λ = 1 – significance level
  - degrees of freedom (df) = number of samples N minus number of parameters estimated

SIGNIFICANCE LEVEL FOR TWO-TAILED TEST						
df	.20	.10	.05	.02	.01	.001
1	3.078	6.314	12.706	31.821	63.657	636.619
2	1.886	2.920	4.303	6.965	9.925	31.598
3	1.638	2.353	3.182	4.541	5.841	12.941
4	1.533	2.132	2.776	3.747	4.604	8.610
5	1.476	2.015	2.571	3.365	4.032	6.859
10	1.372	1.812	2.228	2.764	3.169	4.587
20	1.325	1.725	2.086	2.528	2.845	3.850
30	1.310	1.697	2.042	2.457	2.750	3.646
40	1.303	1.684	2.021	2.423	2.704	3.551
60	1.296	1.671	2.000	2.390	2.660	3.460
120	1.289	1.658	1.980	2.358	2.617	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.291



UAV Example…

- Lets get back to our AUV's GPS measurements of longitude. Here are N measurements:

- The corresponding sample mean, sample standard deviation, and estimated standard error can be calculated:

$$\bar{x} = -120.626368 \quad S = 7.71967E\text{-}06 \quad ESE = 3.8265E\text{-}07$$

Confidence in Measurements

- Examples…. ($\bar{x}$ = *120.626368° , ESE = 3.8265E-07, λ = ESE*t)*

N	P	t	λ
3	95%	4.303	1.7E-06
60	95%	2	7.7E-07
3	99%	9.925	3.8E-06
60	99%	2.66	1.0E-06

SIGNIFICANCE LEVEL FOR TWO-TAILED TEST

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Confidence in Measurements

- Summary:
  1. Calculate your mean $\overline{x}$
  2. Calculate your estimated standard error ESE
  3. For a given df and significance level = $1 - P$, find t from table
  4. Calculate $\lambda = ESE * t$

Outline

- Confidence in Measurements
- Linear Regression
- Error Propagation
- Quantization Error

Linear Regression

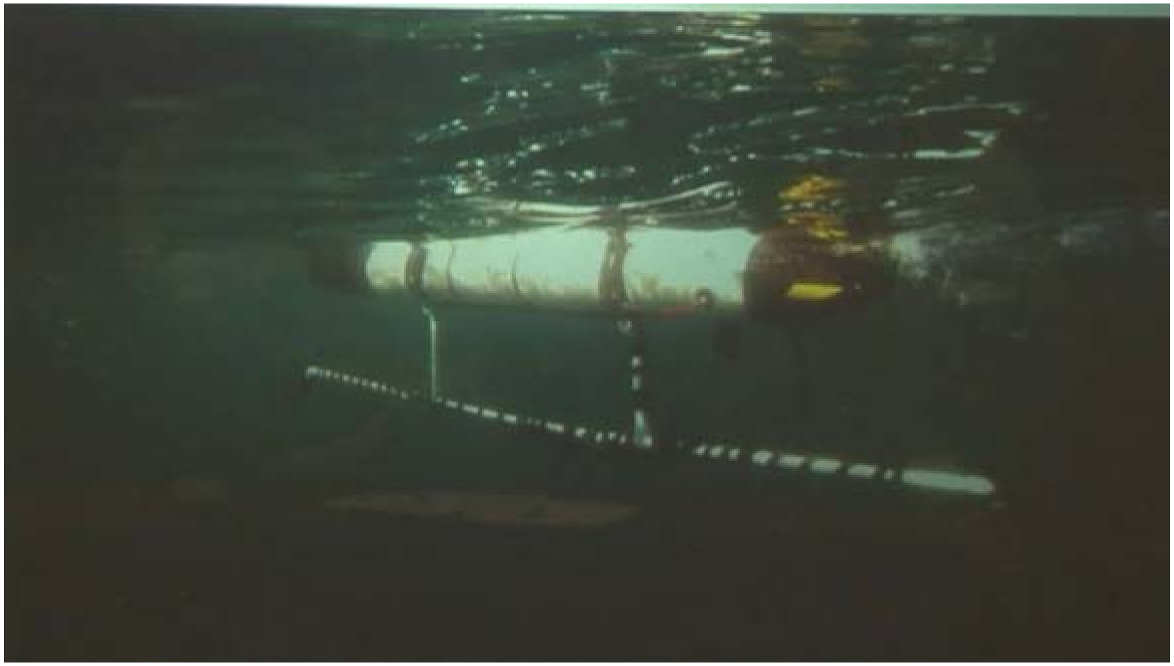
- Sometimes we measure one variable x, but are interested in another variable $y = f(x)$

- Often, $f()$ is assumed to be linear

$$y = \beta_0 + \beta_1 x$$

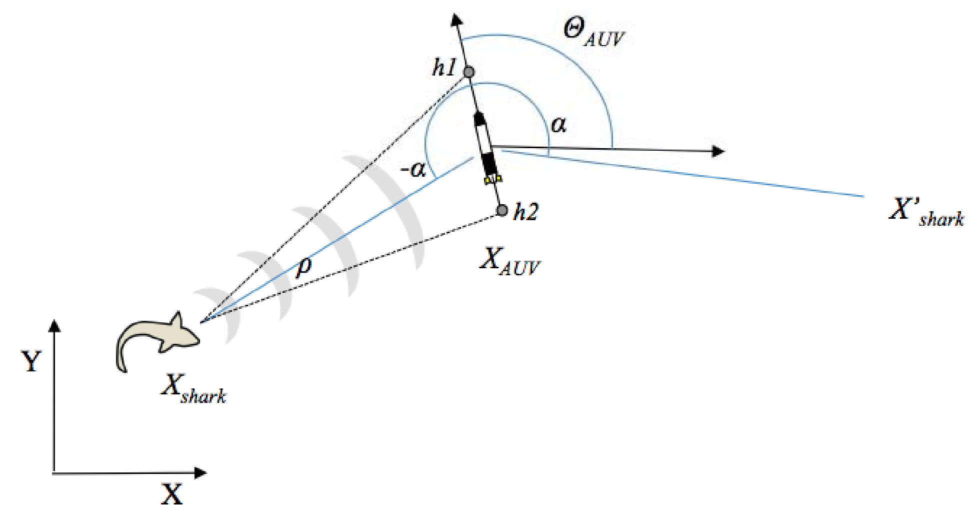
Linear Regression

- Shark tracking Example…

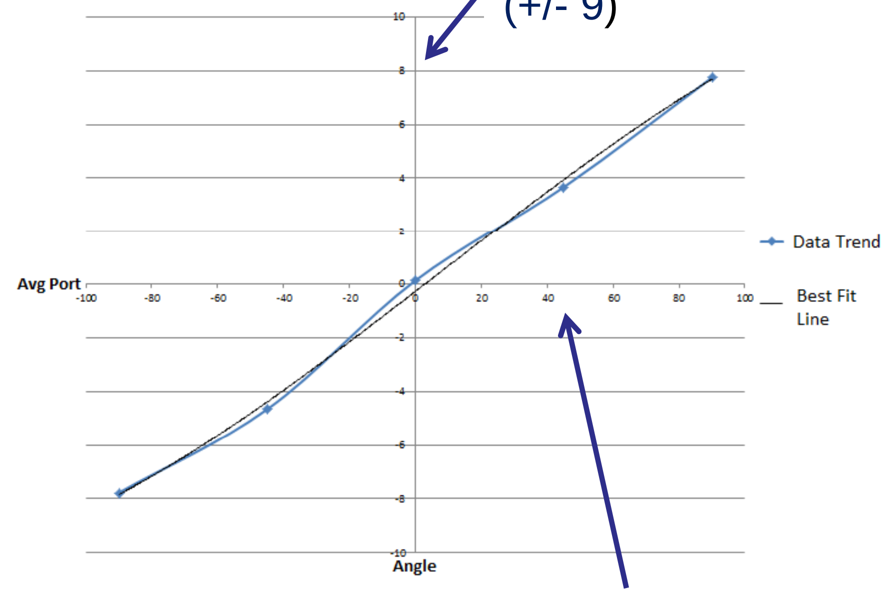


Linear Regression

- Shark tracking Example, sensor calibration



Sensor output (+/- 9)



Angle

Linear Regression

- We usually must estimate the coefficients β_0 and β_1. from a data set:

$$(x_1, y_1), (x_2, y_2), \ldots (x_N, y_N)$$

- Our model becomes

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Linear Regression

- To estimate β_0 and β_1 we minimize the Sum of Squared Errors:

$$SSE = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} \left[y_i - \left(\hat{\beta}_0 + \hat{\beta}_1 x_i \right) \right]^2$$

- This minimization results in

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x} \qquad \hat{\beta}_1 = \frac{\sum_{i=1}^{N} (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{N} (x_i - \overline{x})^2}$$

Linear Regression

- How much confidence do we have in $\hat{\beta}_0$ and $\hat{\beta}_1$?
- Equivalent of the sample standard deviation for linear regression S is the Root Mean Squared Residual, S_e

$$S_e = \sqrt{\frac{SSE}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N} e_i^2}{N-2}} .$$

Linear Regression

- How much confidence do we have in $\hat{\beta}_0$ and $\hat{\beta}_1$?
  - Sample standard error given by…

$$S_{\beta_0} = S_e \sqrt{\frac{1}{N} + \frac{\overline{x}^2}{\sum_{i=1}^{N}(x_i - \overline{x})^2}}. \qquad S_{\beta_1} = S_e \sqrt{\frac{1}{\sum_{i=1}^{N}(x_i - \overline{x})^2}}.$$

(S_e = root mean squared residual)

- *Confidence Intervals*

$$\lambda_{\beta o} = t\, S_{\beta o} \qquad \lambda_{\beta 1} = t\, S_{\beta 1}$$

Linear Regression

- How much confidence do we have in y ?
  - Sample Standard Error given by:

$$S_y = S_e \sqrt{\frac{1}{N} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{N}(x_i-\bar{x})^2}}$$

$$\lambda_y = t\,S_y$$

Linear Regression

- Quick Summary:
  - Given a set of (x, y) pairs, we can calculate

1. The coefficient estimates $\hat{\beta}_0, \hat{\beta}_1$ of the linear regression
2. The confidence limits $\lambda_{\beta 0}, \lambda_{\beta 1}$ on the coefficients
3. The confidence limits λ_y on the y values

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Error Propagation

- Given a function $F(x, y, z, \ldots)$, and known error in variables $x, y, z, \ldots$, what is the error in F ?

Error Propagation

- Assuming that errors are small and the residuals are a reasonable approximation of the errors,
- One can do a Taylor series expansion of F about the true values of the variables, keeping only 1st order terms

$$F - F_{true} = \frac{\partial F}{\partial x}\left(x - x_{true}\right) + \frac{\partial F}{\partial y}\left(y - y_{true}\right) + \frac{\partial F}{\partial z}\left(z - z_{true}\right) + \cdots.$$

Error Propagation

- For errors $\varepsilon = x - x_{true}$, that are systematic, known, and small (so that linear approximations are accurate), we can rewrite as:

$$\varepsilon_F = \frac{\partial F}{\partial x}\varepsilon_x + \frac{\partial F}{\partial y}\varepsilon_y + \frac{\partial F}{\partial z}\varepsilon_z + \cdots.$$

Error Propagation

- If errors of $x, y, z,$ ... are independent random variables (more common), then standard errors are assumed related by root-sum-of-squares:

$$\varepsilon_F = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \varepsilon_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \varepsilon_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \varepsilon_z^2 + \cdots}$$

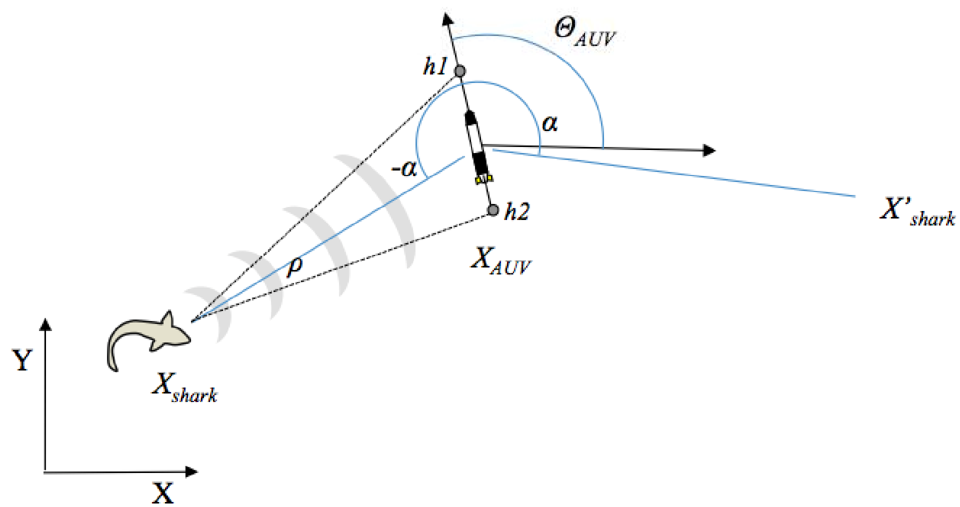
Error Propagation

- Example:
  - We model the range to a shark tag ρ as a function of the strength of the received acoustic signal s.

$$\rho = K_s s^a$$

where

$a < 1$ is constant



Error Propagation

- Example cont':
  - If we know the sample variance S_s^2 in signal strength measurements, and the variance S_K^2 in K_s, we can calculate the corresponding variance in range S_ρ^2

$$S_\rho^2 = (d\rho / ds)^2 S_s^2 + (d\rho / dK_s)^2 S_K^2$$
$$= (aK_s s^{a-1})^2 S_s^2 + (s^a)^2 S_K^2$$

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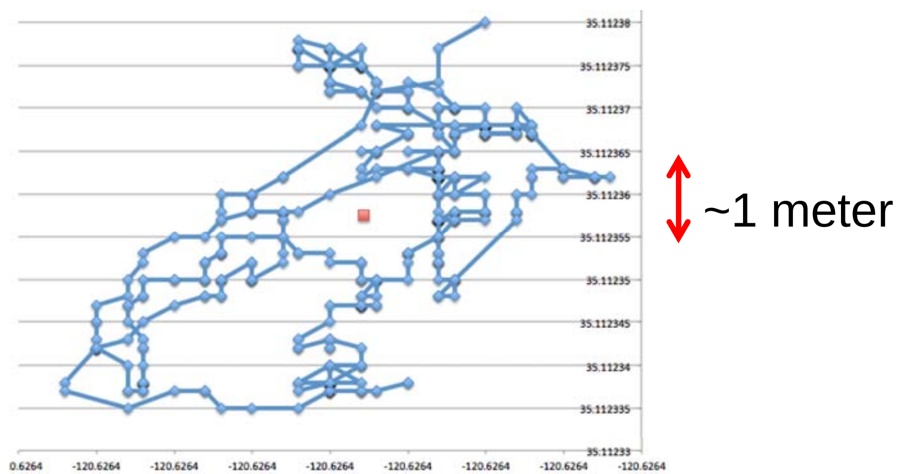
Quantization Error

- Lets revisit the static AUV plot of positions…



We have ~0.1 meter resolution

Quantization Error

- We often witness finite precision in our sensors.
  - If the sample standard deviation S of our measurements is much larger than the quantization error S_q (i.e., $S > 10S_q$), we can ignore the quantization error.
  - If the quantization error is comparable to the sample standard deviation of the measurement (i.e., $S_q < S < 10S_q$), *then we need to include its effects on the error.*
    - $S^2_{used} = S^2 + S^2_q$
  - If the sample standard deviation of the measurement is less than the quantization error (i.e., $S < S_q$), then for the purposes of E80 report the error as $\pm q/2$ (and refer to statistics texts for more accurate treatment)

Quantization Error

- DMM Example:
  - For a 12 bit DAQ, set to +/- 5V, the smallest resolvable voltage, or quantization range, equals the range divided by number of distinct values:

$$q = 10V * 1/2^{12} = 0.027V$$

  - The uncertainty in a DMM is typically 1 least significant digit, and the uncertainty in a given measurement is q±2.
  - For a series of measurements, the standard deviation s_q q/sqrt(12)
  - In an individual voltage measurement, s_q, is ½ the Least significant bit, $s_q = \pm 0.027/2 = \pm 0.013$ V
  - The standard deviation of measurements within q is

Summary

- We can calculate confidence intervals for parameters being measured
- We can construct linear models relating two parameters, along with their confidence intervals
- We can approximate how the error of one parameter affects a function of that parameter
- We can check that the quantization error is insignificant